# Integration using invariant operators: Conformally flat radiation metrics

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Abstract. A new method is presented for obtaining the general conformally flat radiation metric by using the differential operators of Machado Ramos and Vickers (a generalisation of those of Geroch, Held and Penrose) which are invariant under null rotations and rescalings. The solution is found by constructing involutive tables of these derivatives applied to the quantities which arise in the Karlhede classification of this class of metrics.

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#### 1. Introduction

In a recent paper [1] Machado Ramos and Vickers introduced some new operators which are invariant under null rotations. In a subsequent paper [2] this was generalised to incorporate spin and boost transformations so that the resulting formalism depends only upon a choice of a single null direction. Not surprisingly this formalism combines many features of the GHP [3] and null rotation invariant formalisms. In this new formalism the role of the spin coefficients  $\kappa$ ,  $\sigma$ ,  $\rho$  and  $\tau$  is taken up by spinor quantities K, S, R and T given by

$$egin{align} \mathbf{K} &= \kappa \ &\mathbf{S}_{A'} = \sigma ar{\mathbf{o}}_{A'} - \kappa ar{m{\iota}}_{A'} \ &\mathbf{R}_A = 
ho \mathbf{o}_A - \kappa m{\iota}_A \ &\mathbf{T}_{AA'} = au \mathbf{o}_A ar{\mathbf{o}}_{A'} - 
ho \mathbf{o}_A ar{m{\iota}}_{A'} - \sigma m{\iota}_A ar{\mathbf{o}}_{A'} + \kappa m{\iota}_A ar{m{\iota}}_{A'} \end{aligned}$$

Under a transformation of the spin frame given by

$$\mathbf{o}^A \mapsto \lambda \mathbf{o}^A \qquad \boldsymbol{\iota}^A \mapsto \lambda^{-1} \boldsymbol{\iota}^A + \bar{a} \mathbf{o}^A$$

these transform as

$$\mathbf{K} \mapsto \lambda^3 \bar{\lambda} \mathbf{K}$$
 $\mathbf{S}_{A'} \mapsto \lambda^3 \mathbf{S}_{A'}$ 
 $\mathbf{R}_A \mapsto \lambda^2 \bar{\lambda} \mathbf{R}_A$ 
 $\mathbf{T}_{AA'} \mapsto \lambda^2 \mathbf{T}_{AA'}$ 

They are therefore invariant under null rotations and have weight  $\{\mathbf{p}, \mathbf{q}\}$  under spin and boost transformations given by

 $\mathbf{K}:\quad \{3,1\}$ 

 $S: \{3,0\}$ 

 $\mathbf{R}:\quad \{\mathbf{2},\mathbf{1}\}$ 

 $\mathbf{T}:\quad \{\mathbf{2},\mathbf{0}\}$ 

The role of the differential operators  $\mathcal{P}$ ,  $\partial$ ,  $\mathcal{P}'$  and  $\partial'$  is played by new differential operators  $\mathbf{P}$ ,  $\partial$ ,  $\mathbf{P}'$  and  $\partial'$  which act on properly weighted symmetric spinors to produce symmetric spinors of different valence and weight. These operators may all be defined in terms of an auxiliary differential operator  $\mathcal{D}_{ABA'B'}$  which is defined by

$$\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1...C_NC'_1...C'_{N'}} = \mathbf{o}_A\bar{\mathbf{o}}_{A'}\nabla_{BB'}\boldsymbol{\eta}_{C_1...C_NC'_1...C'_{N'}}$$

$$-\left(\mathbf{p}\bar{\mathbf{o}}_{A'}\nabla_{BB'}\mathbf{o}_A + \mathbf{q}\mathbf{o}_A\nabla_{BB'}\bar{\mathbf{o}}_{A'}\right)\boldsymbol{\eta}_{C_1...C_NC'_1...C'_{N'}}$$

$$(1)$$

where  $\eta$  has weight  $\{\mathbf{p}, \mathbf{q}\}$ .

The original motivation for introducing such operators was to improve on the derivative bounds for the Karlhede classification. However it was also hoped that such a formalism would prove useful in finding solutions to Einstein's equations which are invariant under null rotations. The use of the GHP formalism to find exact solutions was pioneered by Held [4,5,6] and in the past few years has been applied by a number of authors including Edgar and Ludwig [7,8,9]. In particular in [9] they demonstrated how the GHP formalism could be used to obtain the complete class of conformally flat radiation metrics. Their method consisted of manipulating the complete system of GHP equations until they were reduced to a complete involutive set of tables for the action of the four GHP operators on four functionally independent  $\{0,0\}$  weighted real scalars, and on one non-trivially weighted complex scalar. As Held had emphasised once such a 'complete set of tables' was obtained the problem was solved; by considering as coordinates the four real  $\{0,0\}$  weighted scalars it was straightforward to write down, directly from their respective tables, the tetrad and hence the metric explicitly.

Another approach to the construction of exact solutions which was originally suggested by Karlhede and Lindström [10], is to apply the techniques used in classifying equivalent metrics in reverse and construct a geometry from a set of elements representing the Riemann tensor and some of its covariant derivatives. This is indeed possible provided certain integrability conditions are satisfied [11, 12]. Furthermore in a number of papers Bradley and Marklund [13, 14] have actually used the method to construct a class of locally rotationally symmetric perfect fluid spacetimes. In the present paper we will combine the ideas used in these two approaches by performing the integration using a systematic application of the commutators of the invariant differential operators to the functional information obtained at each order of the Karlhede classification.

Since the formalism of Machado Ramos and Vickers is invariant under null rotations it should, in principal, be ideally suited to describing the conformally flat radiation metrics. However since the formalism involves symmetric spinors rather than scalars, it was not completely clear how one would carry out the integration in practice. In this paper we demonstrate how the formalism may be used to find all the metrics in the class. The key to the method is to extract  $\{0,0\}$  weighted scalars from spinor quantities. This can happen in two possible ways.

Firstly if one has a spinor field  $\eta_A$  of weight  $\{1,0\}$  such that  $\eta_A \mathbf{o}^A = 0$  then  $\eta_A$  has the form

$$\eta_A = \eta \mathbf{o}_A$$

for some scalar field  $\eta$ . Since  $\mathbf{o}_A$  has weight  $\{1, \mathbf{0}\}$  then  $\eta$  must be a weight  $\{0, 0\}$  scalar. This procedure works for a general type (N, N')-spinor field  $\boldsymbol{\eta}$  of weight  $\{\mathbf{N}, \mathbf{N}'\}$  with the property that  $\boldsymbol{\eta} \cdot \mathbf{o} = 0$  since one must then have

$$oldsymbol{\eta}_{A_1...A'_{N'}} = \eta oldsymbol{o}_{A_1} \ldots oldsymbol{o}_{A_N} ar{oldsymbol{o}}_{A'_1} \ldots ar{oldsymbol{o}}_{A'_{N'}}$$

where  $\eta$  is a weight  $\{0,0\}$  scalar.

The second way in which one can extract scalar fields, is to identify in some invariant way a weight  $\{-1,0\}$  spinor field  $\lambda_A$  which is not proportional to  $\mathbf{o}_A$ , i.e for which  $\lambda_A \mathbf{o}^A = -\lambda \neq 0$ . In this case one may define  $\iota_A$  by

$$\boldsymbol{\lambda}_A = \lambda \boldsymbol{\iota}_A$$

Furthermore one may then contract any other spinor field with  $\iota^A$  to construct invariant scalar fields.

For a conformally flat radiation solution one may choose  $\mathbf{o}_A$  to be aligned with the propagation direction of the radiation and we shall see that this means that the Ricci spinor  $\mathbf{\Phi}_{ABA'B'}$  takes the form

$$\mathbf{\Phi}_{ABA'B'} = \Phi \mathbf{o}_A \mathbf{o}_B \bar{\mathbf{o}}_{A'} \bar{\mathbf{o}}_{B'}$$

where  $\Phi$  is a (real) scalar field of weight  $\{-2, -2\}$ . Other scalar fields may be obtained by taking the covariant derivatives of  $\Phi_{ABA'B'}$ . It turns out that the components of  $\Phi_{AA'BB':CC'}$  may be given in terms of  $P\Phi$ ,  $\partial\Phi$ ,  $\partial\Phi$  and  $P'\Phi$ . Because of the contracted Bianchi identities one finds that  $p\Phi = 0$  but that  $\partial \Phi$  is a type  $\{-1, -3\}$  scalar which despite appearances does not depend upon the choice of  $\iota_A$  (in fact the Bianchi identities show that  $\partial \Phi = \tau \Phi$ ). P' $\Phi$  on the other hand does depend upon the choice of  $\iota_A$  and generically one can choose  $\iota_A$  to make  $p'\Phi$  vanish. Thus for such metrics the curvature and its first derivative provide two scalar fields;  $\Phi$  and  $\partial \Phi$  which are invariant under null rotations. However these quantities have non-trivial weights and are not invariant under spin and boost transformations. We therefore construct an algebraic combination of  $\Phi$  and  $\partial \Phi$ , which we denote A, which is a real scalar field of  $\{0,0\}$  weight and hence invariant under both null rotations and spin and boosts. The remaining two pieces of information in  $\Phi$  and  $\partial \Phi$  may be encoded in a complex field P, of modulus one half, which has spin weight one and boost weight zero and a real field Q which has boost weight one and spin weight zero. For convenience we may combine these into a single complex field  $\bar{P}Q$ .

In the usual Karlhede approach to the classification one starts by choosing the spin frame so that  $\Phi = 1$ . Since  $\Phi$  is real this amounts to fixing the boost freedom in  $\mathbf{o}_A$ . At the next stage the new functional information is provided by the complex scalar field  $\tau$ . The rotational freedom of  $\mathbf{o}_A$  may now be fixed, for instance by demanding the  $\tau$  is real, and the real valued scalar field that one obtains provides the first piece of functional information. The only other information one obtains from the first derivative is contained in  $\mathbf{p}'\Phi$ , but as we have seen above this may be set to zero by partially fixing the null rotation freedom. Thus generically at first order one has one piece of functional information. However in the present approach we wish to work with operators which are invariant under both spin and boosts and null rotations. Thus rather than fixing the gauge by demanding canonical forms for  $\Phi$  and its covariant

derivative we will instead use the information contained in them to construct A, P and Q. We regard P and Q as gauge fields, as the final answer does not depend upon them, while the invariant scalar field A contains the functional information. In the same way we will not fix the null rotation freedom by putting the first derivative into a canonical form, but instead, introduce in a natural way a spinor field  $\mathbf{I}_A$  of weight  $\{-1,0\}$  which satisfies  $\mathbf{o}_A \mathbf{I}^A = 1$ . Again this spinor field contains only gauge information and the final answer does not depend upon it. This should be contrasted with the usual NP integration procedure where one uses the spin coefficients and their derivatives to completely fix the frame, or the GHP integration procedure where one fixes the null rotation freedom.

The procedure then is to manipulate the complete set of spinor equations in the formalism of Machado Ramos and Vickers in an analogous manner to that followed in the GHP formalism in [9]. In general this will involve weighted spinor fields, but by using  $\bar{P}Q$ , and  $\mathbf{I}_A$  (and their conjugates) one can extract  $\{0,0\}$  weighted scalar fields. Eventually one obtains a 'complete set of tables', and from the tables for the four functionally independent  $\{0,0\}$  weighted real scalar fields the tetrad, and hence the metric, may be obtained in exactly the same way as in [9]. Although no new exact solutions are produced, this paper demonstrates in Sections 3 and 4 how to carry out an integration procedure using the new invariant operators adapted to the symmetry of the problem. Furthermore our method involves constructing the geometry solely from the Ricci tensor and its derivatives so that the integration procedure essentially carries out the techniques used in classifying equivalent metrics in reverse. In particular it is clear that the functionally independent scalars, which take on the role of coordinates in our integration procedure, are simply the invariant scalars required by the Karlhede classification. This relationship is discussed in Section 5.

#### 2. The differential operators and the commutators.

We begin with an examination of some of the properties of the differential operators. In particular we need to know the result of contracting  $p'\eta$  with o and  $\bar{o}$ . We start by rewriting equation (1) in the form

$$\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}} = (\mathbf{p}'\boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}})\mathbf{o}_{A}\mathbf{o}_{B}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'}$$

$$- (\partial'\boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}})\mathbf{o}_{A}\mathbf{o}_{B}\bar{\mathbf{o}}_{A'}\bar{\boldsymbol{\iota}}_{B'} - (\partial\boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}})\mathbf{o}_{A}\boldsymbol{\iota}_{B}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'}$$

$$- (\mathbf{p}\boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}})\mathbf{o}_{A}\boldsymbol{\iota}_{B}\bar{\mathbf{o}}_{A'}\bar{\boldsymbol{\iota}}_{B'} + (\mathbf{p}\boldsymbol{\iota}_{A}\bar{\mathbf{o}}_{A'}\mathbf{T}_{BB'} + \mathbf{q}\mathbf{o}_{A}\bar{\boldsymbol{\iota}}_{B'}\bar{\mathbf{T}}_{B'B})\boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}}$$

where P',  $\partial'$ ,  $\partial$  and P are the ordinary GHP operators applied to spinors. The new operators are obtained by contraction with  $\mathbf{o}$  and  $\bar{\mathbf{o}}$ , and symmetrizing.

$$(\mathbf{p}\boldsymbol{\eta})_{AC_{1}...C_{N}A'C'_{1}...C'_{N'}} = \sum_{sym} \mathbf{o}^{B} \bar{\mathbf{o}}^{B'} \mathcal{D}_{ABA'B'} \boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}}$$

$$(\boldsymbol{\partial}\boldsymbol{\eta})_{AC_{1}...C_{N}A'B'C'_{1}...C'_{N'}} = \sum_{sym} \mathbf{o}^{B} \mathcal{D}_{ABA'B'} \boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}}$$

$$(\boldsymbol{\partial}'\boldsymbol{\eta})_{ABC_{1}...C_{N}A'C'_{1}...C'_{N'}} = \sum_{sym} \bar{\mathbf{o}}^{B'} \mathcal{D}_{ABA'B'} \boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}}$$

$$(\boldsymbol{p}'\boldsymbol{\eta})_{ABC_{1}...C_{N}A'B'C'_{1}...C'_{N'}} = \sum_{sym} \mathcal{D}_{ABA'B'} \boldsymbol{\eta}_{C_{1}...C_{N}C'_{1}...C'_{N'}}$$

where  $\sum_{sym}$  indicates symmetrization over all free primed and unprimed indices.

In the case of a scalar field this gives

$$(\mathbf{P}'\eta)_{ABA'B'} = (\mathbf{P}'\eta)\mathbf{o}_{A}\mathbf{o}_{B}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - (\partial'\eta - q\bar{\tau}\eta)\mathbf{o}_{A}\mathbf{o}_{B}\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B'}) - (\partial\eta - p\tau\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} + (\mathbf{P}\eta - p\rho\eta - q\bar{\rho}\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B'}) + (p\kappa\boldsymbol{\iota}_{A}\boldsymbol{\iota}_{B}\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B'}) + q\bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'} - p\sigma\boldsymbol{\iota}_{A}\boldsymbol{\iota}_{B}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - q\bar{\sigma}\mathbf{o}_{A}\mathbf{o}_{B}\boldsymbol{\iota}_{A'}\boldsymbol{\iota}_{B'})\eta$$
(2)

$$(\partial' \eta)_{ABA'} = (\partial' \eta) \mathbf{o}_A \mathbf{o}_B \bar{\mathbf{o}}_{A'} - (\mathbf{p}\eta - p\rho\eta) \mathbf{o}_{(A} \boldsymbol{\iota}_{B)} \bar{\mathbf{o}}_{A'} + (q\bar{\sigma} \mathbf{o}_A \mathbf{o}_B \boldsymbol{\iota}_{A'} - p\kappa \boldsymbol{\iota}_A \boldsymbol{\iota}_B \bar{\mathbf{o}}_{A'} - q\bar{\kappa} \mathbf{o}_{(A} \boldsymbol{\iota}_B) \bar{\boldsymbol{\iota}}_{A'}) \eta$$
(3)

$$(\partial \eta)_{AA'B'} = (\partial \eta) \mathbf{o}_{A} \bar{\mathbf{o}}_{A'} \bar{\mathbf{o}}_{B'} - (\bar{\mathbf{p}} \eta - q \bar{\rho} \eta) \mathbf{o}_{A} \bar{\mathbf{o}}_{(A'} \bar{\boldsymbol{\iota}}_{B')} + (p \sigma \boldsymbol{\iota}_{A} \bar{\mathbf{o}}_{A'} \bar{\mathbf{o}}_{B'} - p \kappa \boldsymbol{\iota}_{A} \bar{\mathbf{o}}_{(A'} \bar{\boldsymbol{\iota}}_{B')} - q \bar{\kappa} \mathbf{o}_{A} \bar{\boldsymbol{\iota}}_{A'} \bar{\boldsymbol{\iota}}_{B'}) \eta$$

$$(4)$$

$$(\mathbf{p}\eta)_{AA'} = (\mathbf{p}\eta)\mathbf{o}_{A}\bar{\mathbf{o}}_{A'} + (p\kappa\boldsymbol{\iota}_{A}\bar{\mathbf{o}}_{A'} - q\bar{\kappa}\mathbf{o}_{A}\bar{\boldsymbol{\iota}}_{A'})\eta \tag{5}$$

Contracting (2) with  $\bar{\mathbf{o}}^{B'}$  gives

$$(\mathbf{p}'\eta)_{ABA'B'}\bar{\mathbf{o}}^{B'} = \frac{1}{2}\{(\boldsymbol{\partial}'\eta)_{ABA'} - q(\bar{\tau}\mathbf{o}_{A}\mathbf{o}_{B}\bar{\mathbf{o}}_{A'} - \bar{\rho}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\mathbf{o}_{A'} - \bar{\sigma}\mathbf{o}_{A}\mathbf{o}_{B}\bar{\boldsymbol{\iota}}_{A'} + \bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\boldsymbol{\iota}_{A'})\eta\}$$
$$= \frac{1}{2}\{(\boldsymbol{\partial}'\eta)_{ABA'} - q\bar{\mathbf{T}}_{A'(A}\mathbf{o}_{B)}\eta\}$$

or in the compacted notation

$$(\mathbf{p}'\eta) \cdot \bar{\mathbf{o}} = \frac{1}{2} \{ (\mathbf{\partial}'\eta) - q\bar{\mathbf{T}}\eta \}$$
 (6)

Similar calculations give

$$(\mathbf{P}'\eta) \cdot \mathbf{o} = \frac{1}{2} \{ (\partial \eta) - p\mathbf{T}\eta \}$$
 (7)

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$$(\partial' \eta) \cdot \mathbf{o} = \frac{1}{2} \{ (\mathbf{p} \eta) - p \mathbf{R} \eta \}$$
 (8)

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$$(\partial \eta) \cdot \bar{\mathbf{o}} = \frac{1}{2} \{ (\mathbf{p} \eta) - q \bar{\mathbf{R}} \eta \} \tag{9}$$

and

$$(\mathbf{p}'\eta) \cdot \mathbf{o} \cdot \bar{\mathbf{o}} = \frac{1}{4} \{ (\mathbf{p}\eta) - p\mathbf{R}\eta - q\bar{\mathbf{R}}\eta \}$$
(10)

If  $\eta$  is a spinor the above contractions become more complicated. For example for a type (1,0)-spinor  $\eta_A$  of weight  $\{\mathbf{p},\mathbf{q}\}$  we get

$$(\mathbf{p}'\boldsymbol{\eta})\cdot\mathbf{o} = \frac{1}{3}\{\mathbf{p}'(\boldsymbol{\eta}\cdot\mathbf{o}) + (\boldsymbol{\partial}'\boldsymbol{\eta}) - (\mathbf{p}-1)\mathbf{T}\boldsymbol{\eta}\}$$

Although the definition of the differential operators is quite complicated, the fact that they take symmetric spinors to symmetric spinors means that one can write down the equations in an index free notation.

The Ricci equations, Bianchi equations and the commutators for the general case are given in [2]. This complete system of equations is completely equivalent to Einstein's equations, and to find solutions to Einstein's equations this system will therefore have to be completely integrated. This complete system also contains exactly the same information as the analogous complete systems in the NP and GHP formalisms respectively. However, in view of the more complicated nature of the operators in this formalism, some of the information which resided in the Ricci equations in NP and/or GHP formalisms is contained within the commutators in this formalism; in particular these commutators contain inhomogeneous terms explicitly dependent on the weight and valence of spinor on which they act. To extract all the information in the commutators we need to apply them to, [15]

- (i) four functionally independent {0,0} weighted real scalars,
- (ii) one  $\{p,q\}$  weighted complex scalar where  $p \neq \pm q$ ,
- (iii) one valence (1,0) spinor,  $\mathbf{I}_A$  of weight  $\{-1,0\}$ .

Of course, we can extract all the information by applying the commutators to different (but essentially equivalent) combinations of these scalars and spinor; however the particular choices above are best suited to our integration procedure since the four  $\{0,0\}$  weighted real scalars will become the coordinates, the complex scalar is given by the gauge field  $\bar{P}Q$ , while the spinor  $\mathbf{I}_A$  will be identified with the second dyad spinor  $\mathbf{\iota}_A$ .

For the special case of the conformally flat solutions we are considering here there is considerable simplification, particularly in the Ricci and Bianchi equations. Choosing  $\mathbf{o}_A$  to be aligned with the propagation direction of the radiation means that the Ricci spinor takes the form

$$\mathbf{\Phi}_{ABA'B'} = \Phi \mathbf{o}_A \mathbf{o}_B \bar{\mathbf{o}}_{A'} \bar{\mathbf{o}}_{B'}$$

where  $\Phi$  is a real scalar field of weight  $\{-2, -2\}$ . All the other components of the curvature vanish. Substituting into the Bianchi identities in this formalism ((55)-(65) of reference [2]) gives

$$\mathbf{K} = 0 \tag{11}$$

$$\mathbf{S} = 0 \tag{12}$$

$$\mathbf{R} = 0 \tag{13}$$

together with

$$\mathbf{p}\mathbf{\Phi}_{22} = 0 \tag{14}$$

$$\partial' \Phi_{22} = \bar{\mathbf{T}} \Phi_{22} \tag{15}$$

Equations (11)–(13) are of course equivalent to the GHP equations  $\kappa = \sigma = \rho = 0$  so that **T** has the form

$$\mathbf{T}_{AA'} = \tau \mathbf{o}_A \bar{\mathbf{o}}_{A'}$$

and (14) and (15) then give the GHP equations

$$b\Phi = 0$$

$$\bar{\partial}\Phi = \bar{\tau}\Phi$$

Most of the Ricci equations are identically satisfied, with the remaining equations being

$$\mathbf{p}\mathbf{T} = 0 \qquad \qquad \mathbf{p}\tau = 0$$

$$\mathbf{\partial}\mathbf{T} = \mathbf{T}^2 \qquad \Leftrightarrow \qquad \partial\tau = \tau^2$$

$$\mathbf{\partial}'\mathbf{T} = \mathbf{T}\bar{\mathbf{T}} \qquad \qquad \partial'\tau = \tau\bar{\tau}$$
(16)

Finally the commutators applied to a general symmetric spinor  $\eta$  reduce to

$$[\mathbf{p}', \mathbf{p}]\boldsymbol{\eta} = -\bar{\mathbf{T}}\boldsymbol{\partial}\boldsymbol{\eta} - \mathbf{T}\boldsymbol{\partial}'\boldsymbol{\eta}$$

$$[\mathbf{p}', \boldsymbol{\partial}']\boldsymbol{\eta} = -\bar{\mathbf{T}}\mathbf{p}'\boldsymbol{\eta} - \boldsymbol{\Phi}_{22}(\boldsymbol{\eta} \cdot \bar{\mathbf{o}})$$

$$[\mathbf{p}', \boldsymbol{\partial}]\boldsymbol{\eta} = -\mathbf{T}\mathbf{p}'\boldsymbol{\eta} - \boldsymbol{\Phi}_{22}(\boldsymbol{\eta} \cdot \mathbf{o})$$

$$[\mathbf{p}, \boldsymbol{\partial}']\boldsymbol{\eta} = 0$$

$$[\mathbf{p}, \boldsymbol{\partial}]\boldsymbol{\eta} = 0$$

$$[\mathbf{p}, \boldsymbol{\partial}]\boldsymbol{\eta} = 0$$

$$[\boldsymbol{\partial}', \boldsymbol{\partial}]\boldsymbol{\eta} = 0$$

$$(17)$$

Note that the terms involving a contraction with  $\mathbf{o}$  are absent when such a contraction is impossible (as is the case when  $\eta$  is a scalar).

In this particular case it is obvious that the Bianchi and Ricci equations supply comparatively little information. Therefore most of the information is contained in the commutators, and so it is very important that we apply the commutators in a systematic manner as outlined above, in order to ensure that all possible information is obtained. The Bianchi and Ricci equations only supply us with the respective partial tables for  $\tau$  and  $\Phi$  given above, so, in the next section, we will need to complete these tables (for their  $\mathbf{P}'$  derivative) with two unknown spinors. The commutators will first be applied to a complex weighted scalar  $\bar{P}Q$  formed from  $\tau$  and  $\Phi$ , and next to one of the two unknown spinors which can be scaled to be of the type and weight of  $\mathbf{I}_A$ ; then we will apply the commutators to the  $\{0,0\}$  weighted real scalar A (which is essentially  $\tau\bar{\tau}$ ). These operations will generate new  $\{0,0\}$  weighted scalars and their respective tables; we will then choose three of these real scalars, ensuring that they are functionally independent of A and of each other, and apply the commutators to them. After a little tidying up we will obtain an essentially involutive set of tables for four functionally independent  $\{0,0\}$  weighted real scalars, for one weighted complex scalar and for the spinor  $\mathbf{I}_A$ .

By considering the four  $\{0,0\}$  weighted real scalars as coordinates, we can write down, directly from their tables, the metric. The table for the weighted scalar will supply explicitly the 'badly weighted' spin coefficients  $\alpha$ ,  $\beta$ ,  $\epsilon$  and  $\gamma$ , while the table for  $\mathbf{I}_A$  yields the other 'missing' spin coefficients  $\mu$ ,  $\nu$ ,  $\lambda$  and  $\pi$ ; of course, our interest is usually just in obtaining the metric, and so we would not normally bother evaluating these other spin coefficients explicitly. However although the information in these last two tables is not used directly in the calculation of the metric the tables play an essential role in our intermediate calculations and in generating some of the coordinates and their tables.

#### 3. The integration procedure: the generic case

#### 3.1. Preliminary rearranging

As explained in section 1 the Riemann tensor and its first derivative supply three real scalars which can easily be rearranged to give one real zero-weighted  $(\tau \bar{\tau})$  and two real weighted scalars,  $\Phi$  and  $\arg(\tau/\bar{\tau})$ . However, simply to keep the presentation of subsequent calculations to a minimum, it will be convenient to rearrange slightly these three scalars, and use instead the zero-weighted scalar

$$A = \frac{1}{\sqrt{2\tau\bar{\tau}}}\tag{18}$$

and the weighted scalars

$$P = \sqrt{\frac{\tau}{2\bar{\tau}}} \qquad \text{with} \qquad P\bar{P} = \frac{1}{2} \tag{19}$$

$$Q = \frac{\sqrt{\Phi}}{\sqrt[4]{2\tau\bar{\tau}}} \tag{20}$$

of respective weights P {1, -1} and Q {-1, -1}. (We are assuming  $\tau \neq 0$ , and so each of A, P, Q will always be defined and different from zero.)

These particular choices enable us to replace (16) with the very simple equations

$$\mathbf{p}A = 0$$

$$\mathbf{\partial}A = -P$$

$$\mathbf{\partial}'A = -\bar{P}$$
(21)

and

$$\mathbf{p}P = 0 = \mathbf{p}Q$$

$$\mathbf{\partial}P = 0 = \mathbf{\partial}Q$$

$$\mathbf{\partial}'P = 0 = \mathbf{\partial}'Q$$
(22)

# 3.2. Applying commutators to one complex weighted scalar and to one spinor

For our integration procedure we begin with a table for the weighted scalar  $(\bar{P}Q)$ , whose weight is  $\{-2,0\}$ ,

$$\mathbf{p}(\bar{P}Q) = 0$$

$$\mathbf{\partial}(\bar{P}Q) = 0$$

$$\mathbf{\partial}'(\bar{P}Q) = 0$$

$$\mathbf{p}'(\bar{P}Q) = -\frac{Q}{A}\mathbf{I}$$
(23)

where we have completed the table with a spinor  $\mathbf{I}$ , which is as yet undetermined. (We have introduced the weighted factor  $\frac{-Q}{A}$  in the above definition for  $\mathbf{I}$  simply for convenience in later calculations.)

It follows from (6) and (7) that

$$\mathbf{I} \cdot \bar{\mathbf{o}} = -\frac{A}{Q} (\mathbf{p}'(\bar{P}Q)) \cdot \bar{\mathbf{o}} = -\frac{A}{Q} \mathbf{\partial}'(\bar{P}Q)$$

$$= 0$$
(24)

$$\mathbf{I} \cdot \mathbf{o} = -\frac{A}{Q} (\mathbf{p}'(\bar{P}Q)) \cdot \mathbf{o} = -\frac{A}{Q} (\mathbf{\partial}(\bar{P}Q) + 2\tau(\bar{P}Q))$$

$$= -1$$
(25)

Hence **I** is a (1,0) type spinor, and from

$$(\mathbf{P}'(\bar{P}Q))_{ABA'B'} = -\frac{Q}{A}\mathbf{I}_{(A}\mathbf{o}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'}$$

we conclude that its weight is  $\{-1,0\}$ .

So now we have to apply the commutators to the table for (PQ) which yields a partial table for the spinor **I**; the complete table can be written

$$\mathbf{p}\mathbf{I} = 0$$

$$\mathbf{\partial}\mathbf{I} = 0$$

$$\mathbf{\partial}'\mathbf{I} = 0$$

$$\mathbf{p}'\mathbf{I} = \frac{\bar{P}Q^2}{A}\mathbf{W}$$
(26)

where we have completed the table with a spinor  $\mathbf{W}$ , which is as yet undetermined. It follows that

$$\mathbf{W} \cdot \bar{\mathbf{o}} = \frac{A}{\bar{P}Q^2} (\mathbf{p}' \mathbf{I}) \cdot \bar{\mathbf{o}}$$

$$= 0$$
(27)

$$\mathbf{W} \cdot \mathbf{o} = -\frac{A}{\bar{P}Q^2} (\mathbf{P}'\mathbf{I}) \cdot \mathbf{o}$$

$$= \frac{1}{Q^2 \bar{P}^2} \mathbf{I}$$
(28)

Hence

$$\mathbf{W} = -\frac{1}{2\bar{P}^2 Q^2} \mathbf{I}^2 + W \tag{29}$$

where **W** is a (2,0) type spinor of weight  $\{2,0\}$ , and W is a zero-weighted complex scalar.

We next have to apply the commutators to the table for  $\mathbf{I}$  which yields a partial table for the spinor  $\mathbf{W}$ ; under the substitution (29) we obtain a partial table for the zero-weighted complex scalar W,

$$\mathbf{p}W = 0$$

$$\mathbf{\partial}W = -2P$$

$$\mathbf{\partial}'W = 0$$
(30)

# 3.3. Finding four coordinate candidates, and applying commutators to them

We have obtained complete tables for the weighted scalar  $(\bar{P}Q)$ , and for the spinor I, and applied the commutators to each; so it remains to obtain complete tables for four real zero-weighted scalars, and to apply the commutators to all four of these scalars. Clearly A and W, for which we already have partial tables, are obvious candidates.

The complete table for A can be written

$$\mathbf{p}A = 0$$

$$\mathbf{\partial}A = -P$$

$$\mathbf{\partial}'A = -\bar{P}$$

$$\mathbf{p}'A = \frac{Q}{A}\mathbf{N}$$
(31)

where we have completed the table with a spinor N, which is as yet undetermined. It follows that

$$\mathbf{N} \cdot \bar{\mathbf{o}} = \frac{A}{Q} (\mathbf{P}' A) \cdot \bar{\mathbf{o}} = \frac{A}{Q} \partial' A = -\frac{A}{Q} \bar{P}$$
(32)

$$\mathbf{N} \cdot \mathbf{o} = \frac{A}{Q} (\mathbf{P}' A) \cdot \mathbf{o} = \frac{A}{Q} \partial A = -\frac{A}{Q} P \tag{33}$$

Hence

$$\mathbf{N} = \frac{AP}{Q}\mathbf{I} + \frac{A\bar{P}}{Q}\bar{\mathbf{I}} + N \tag{34}$$

where **N** is a hermitian (1,1) type spinor of weight  $\{1,1\}$ , and N is a zero-weighted real scalar.

We now have to apply the commutators to A, which yields a partial table for  $\mathbf{N}$ ; under the substitution (34) we obtain a partial table for the zero-weighted real scalar N,

$$\mathbf{p}N = -\frac{1}{Q}$$

$$\boldsymbol{\partial}N = \bar{\mathbf{I}}/Q$$

$$\boldsymbol{\partial}'N = \mathbf{I}/Q$$
(35)

Next, considering W, we can write down its complete table,

$$\mathbf{p}W = 0$$

$$\mathbf{\partial}W = -2P$$

$$\mathbf{\partial}'W = 0$$

$$\mathbf{p}'W = \frac{Q}{A}\mathbf{Z}$$
(36)

where we have completed the table with a spinor  $\mathbf{Z}$ , which is as yet undetermined. It follows that

$$\mathbf{Z} \cdot \bar{\mathbf{o}} = \frac{A}{Q} (\mathbf{P}'W) \cdot \bar{\mathbf{o}} = 0 \tag{37}$$

Integration using invariant operators

$$\mathbf{Z} \cdot \mathbf{o} = \frac{A}{Q} (\mathbf{P}'W) \cdot \mathbf{o} = -\frac{2AP}{Q}$$
(38)

Hence

$$\mathbf{Z} = \frac{2AP}{Q}\mathbf{I} + Z \tag{39}$$

where **Z** is a (1,0) type spinor of weight  $\{1,0\}$ , and Z is a zero-weighted complex scalar.

We now have to apply the commutators to W, which yields a partial table for  $\mathbf{Z}$ ; under the substitution (39) we obtain a partial table for the zero-weighted complex scalar Z,

$$\mathbf{p}Z = -\frac{1}{Q}$$

$$\mathbf{\partial}Z = \bar{\mathbf{I}}/Q$$

$$\mathbf{\partial}'Z = \mathbf{I}/Q$$
(40)

Having applied our commutators to (the equivalent of) three real zero-weighted scalars, we need to identify (at least) one more; clearly N — which is real and zero-weight — is the obvious candidate. Using (35) we can write down a complete table for N,

$$\mathbf{P}N = -\frac{1}{Q}$$

$$\boldsymbol{\partial}N = \overline{\mathbf{I}}/Q$$

$$\boldsymbol{\partial}'N = \mathbf{I}/Q$$

$$\mathbf{P}'N = \frac{Q}{A}\mathbf{L}$$
(41)

where we have completed the table with a spinor  $\mathbf{L}$ , which is as yet undetermined. It follows that

$$\mathbf{L} \cdot \bar{\mathbf{o}} = \frac{A}{Q} (\mathbf{p}' N) \cdot \bar{\mathbf{o}} = \frac{A}{Q^2} \mathbf{I}$$
(42)

$$\mathbf{L} \cdot \mathbf{o} = \frac{A}{O}(\mathbf{p}'N) \cdot \mathbf{o} = \frac{A}{O^2}\bar{\mathbf{I}}$$
(43)

Hence

$$\mathbf{L} = -\frac{A}{Q^2}\mathbf{I}\bar{\mathbf{I}} + L \tag{44}$$

where **L** is a hermitian (1,1) type spinor of weight  $\{1,1\}$ , and L is a zero-weighted real scalar.

We now have to apply the commutators to N, which yields a partial table for  $\mathbf{L}$ ; under the substitution (49) we obtain a partial table for the zero-weighted real scalar L,

$$\mathbf{p}L = 0$$

$$\mathbf{\partial}L = P\bar{W}$$

$$\mathbf{\partial}'L = \bar{P}W$$
(45)

So we now have applied our commutators to (the equivalent of) four real zeroweighted scalars, and providing that these scalars are functionally independent, they can be adopted as coordinates. (It will be easier to check for this functional independence from the scalar operator form of these tables.) Furthermore, we have now obtained in an explicit form all the information about this class of spaces.

However, our tables for the zero-weighted scalars A, W, N are not completely involutive with respect to these zero-weighted scalars, since they contain also the zero-weighted scalar functions L, Z; but we also know the defining constraints (45), (40) on these functions. In the next subsection we will rearrange these tables a little in order that the defining constraints on these extra scalars have a particularly simple form.

# 3.4. Simpler form of the six tables in spinor operators

Before translating our tables into the scalar operators, it will be convenient to write the complex scalars W, Z respectively in their real and imaginary parts, and by a slight rearranging obtain a simpler presentation of the tables.

Putting

$$M = \frac{1}{2}(W + \bar{W}) - A$$
  $B = \frac{i}{2}(W - \bar{W})$   
 $F = \frac{1}{2}(Z + \bar{Z}) - N$   $E = \frac{i}{2}(Z - \bar{Z})$ 

we replace the table for (complex) W with the two tables,

$$\mathbf{p}M = 0$$

$$\mathbf{\partial}M = 0$$

$$\mathbf{\partial}'M = 0$$

$$\mathbf{p}'M = \frac{QF}{A}$$

$$\mathbf{p}B = 0$$
(46)

$$\mathbf{\partial}B = -iP$$

$$\mathbf{\partial}'B = i\bar{P}$$

$$\mathbf{P}'B = i(P\mathbf{I} - \bar{P}\bar{\mathbf{I}}) + \frac{QE}{A}$$
(47)

The scalars E, F satisfy the simple conditions,

$$\mathbf{p}E = \mathbf{p}F = 0$$

$$\mathbf{\partial}E = \mathbf{\partial}F = 0$$

$$\mathbf{\partial}'E = \mathbf{\partial}'F = 0$$
(48)

Under the substitution

$$L = S - MA - \frac{1}{2}(A^2 + B^2) \tag{49}$$

the table (41) for N becomes

$$\mathbf{p}N = -\frac{1}{Q}$$

$$\boldsymbol{\partial}N = \bar{\mathbf{I}}/Q$$

$$\boldsymbol{\partial}'N = \mathbf{I}/Q$$

$$\mathbf{p}'N = -\frac{1}{Q}\mathbf{I}\bar{\mathbf{I}} + \frac{Q}{A}\left(S - MA - \frac{1}{2}(A^2 + B^2)\right)$$
(50)

while S satisfies the simple conditions

$$\mathbf{p}S = 0$$

$$\mathbf{\partial}S = 0$$

$$\mathbf{\partial}'S = 0$$
(51)

Finally, alongside the tables for the other three coordinate candidates, we include the table (31) for A,

$$\mathbf{p}A = 0$$

$$\mathbf{\partial}A = -P$$

$$\mathbf{\partial}'A = -\bar{P}$$

$$\mathbf{p}'A = P\mathbf{I} + \bar{P}\bar{\mathbf{I}} + \frac{Q}{A}N$$
(52)

For completeness we add the other two tables, although we will not need to use them in obtaining the metric,

$$\mathbf{p}(\bar{P}Q) = 0$$

$$\mathbf{\partial}(\bar{P}Q) = 0$$

$$\mathbf{\partial}'(\bar{P}Q) = 0$$

$$\mathbf{p}'(\bar{P}Q) = -\frac{Q}{A}\mathbf{I}$$
(53)

$$\mathbf{P}\mathbf{I} = 0$$

$$\mathbf{\partial}\mathbf{I} = 0$$

$$\mathbf{\partial}'\mathbf{I} = 0$$

$$\mathbf{P}'\mathbf{I} = -\frac{P}{A}\mathbf{I}^2 + \frac{\bar{P}Q^2}{A}(A + M - iB)$$
(54)

## 3.5. The tables in terms of scalar operators

If we identify the spinor  $\mathbf{I}_A$  with the second dyad spinor  $\boldsymbol{\iota}_A$ , then the four tables for the zero weighted coordinate candidates M, N, A, B can be easily translated into the ordinary Newman-Penrose scalar operators,

$$DM = 0$$

$$\delta M = 0$$

$$\bar{\delta}M = 0$$

$$\Delta M = \frac{QF}{A}$$
(55)

$$DN = -\frac{1}{Q}$$

$$\delta N = 0$$

$$\bar{\delta}N = 0$$

$$\Delta N = \frac{Q}{A} \left( S - MA - \frac{1}{2} (A^2 + B^2) \right)$$
(56)

$$DA = 0$$

$$\delta A = -P$$

$$\bar{\delta} A = -\bar{P}$$

$$\Delta A = \frac{QN}{A}$$
(57)

$$DB = 0$$

$$\delta B = -iP$$

$$\bar{\delta}B = i\bar{P}$$

$$\Delta B = \frac{QE}{A}$$
(58)

We note again that the four tables for the real zero-weighted scalars are not strictly involutive in these scalars; there occur also the three real scalars E, F, S, which satisfy

$$DE = DF = DS = 0$$

$$\delta E = \delta F = \delta S = 0$$

$$\bar{\delta} E = \bar{\delta} F = \bar{\delta} S = 0$$
(59)

The rearranging which we have just carried out was in order to obtain this simple version of these conditions. Clearly  $\nabla E$ ,  $\nabla F$ ,  $\nabla S$ , are all parallel and each is also parallel to  $\nabla M$ ; hence the zero-weighted scalars E, F, S can each be assumed to be an arbitrary function of the coordinate candidate M alone (and independent of the coordinate candidates A, B, N).

# 3.6. Using coordinate candidates as coordinates

If we now make the obvious choice of the coordinate candidates as coordinates

$$m = M, \quad n = N, \quad a = A, \quad b = B \tag{60}$$

the above four tables for the zero-weighted scalars enable us to write down the tetrad vectors in the coordinates m, n, a, b,

$$l^{i} = (0, \frac{-1}{Q}, 0, 0)$$

$$n^{i} = \frac{Q}{a} \left( F, (S - ma - \frac{1}{2}a^{2} - \frac{1}{2}b^{2}), n, E \right)$$

$$m^{i} = P(0, 0, -1, -i)$$

$$\bar{m}^{i} = \bar{P}(0, 0, -1, i)$$
(61)

where E, F, S are arbitrary functions of the coordinate m. The metric follows immediately from the equation

$$g^{ij} = 2l^{(i}n^{j)} - 2m^{(i}\bar{m}^{j)} \tag{62}$$

and we see that it does not depend upon the gauge fields P and Q.

However we noted in the last subsection that this whole procedure is dependent on the condition that the zero-weighted scalars which we choose as coordinate candidates are functionally independent. However, if we make the assumption that none of these scalars are constants, then a check of the determinant formed from the four tables for M, N, A, B shows that all four scalars are functionally independent. Now it is easy to check that none of N, A, B can be constant, but M might be; therefore the tetrad obtained above is not the most general that can be obtained for this class of spacetime. In the next section we will consider the case where M is constant and we need to find a fourth coordinate.

# 4. The integration procedure: the complete case

#### 4.1. Preliminaries

In the previous section we assumed that M was not a constant, so that we were able to choose it as our fourth coordinate candidate. Next, we should look at the excluded case where M is a constant. In such a situation, clearly F is zero, but we still have the possibility of choosing E or S as our fourth coordinate. Once we make such a choice then we could continue in a similar manner as in the last section, building our tables, and hence the tetrad, around our four coordinate candidates. However, if all of the functions M, E, S are constants, then it will not be possible to find the fourth coordinate candidate directly; we emphasise that in such circumstances no additional independent quantities can be generated by any direct manipulations of the tables and the commutators. In such a situation we still need a fourth coordinate candidate in order to extract the remaining information from the commutators. We shall now show that by an indirect approach a fourth coordinate candidate can in fact be found, so that we can obtain the complete metric as one expression.

# 4.2. Finding a fourth coordinate candidate indirectly, and extracting all the information from the complete system.

The results in section 3 up to the end of subsection 3.4 apply; however, when we are considering tables explicitly for our coordinate candidates we consider only the three coordinate candidates N, B, A while the zero-weighted scalar M is not now included as a coordinate candidate.

So, clearly we do not have our full quota of four coordinate candidates, but we do not wish to use any of the remaining quantities from the tables, since it would involve the additional assumption of that quantity being non-constant. However, we know that we have not yet extracted all the information from the commutators (17), since they have only been applied to three zero-weighted coordinate candidates. So we closely examine the structure of the commutators (17) to determine whether they suggest the existence of a fourth zero-weighted scalar, functionally independent of the first three coordinate candidates, whose table is consistent with the commutators. In fact, we get a strong hint from the previous section, and consider the possibility of the existence of a real zero-weighted scalar T, which satisfies the table

$$\mathbf{p}T = 0$$

$$\mathbf{\partial}T = 0$$

$$\mathbf{\partial}'T = 0$$

$$\mathbf{p}'T = Q/A$$
(63)

It is straightforward to confirm that such a choice is consistent with the commutators (17) and creates no inconsistency with the other tables.

### 4.3. The scalar tables.

As in Section 2.5 we identify the spinor  $\mathbf{I}_A$  with the second dyad spinor  $\mathbf{\iota}_A$ , so that the four tables can be translated into the ordinary Newman-Penrose scalar operators. Therefore, this is simply equivalent to replacing the table for M (46), with the table (63) for T,

$$DT = 0$$

$$\delta T = 0$$

$$\bar{\delta}T = 0$$

$$\Delta T = Q/A$$
(64)

while the real zero-weighted quantities E, M, S satisfy

$$DE = DM = DS = 0$$

$$\delta E = \delta M = \delta S = 0$$

$$\bar{\delta} E = \bar{\delta} M = \bar{\delta} S = 0$$
(65)

and so E, M, S are now arbitrary functions of T only. A check on the determinant formed from the four scalar tables (57), (58), (56), (64), for A, B, N, T respectively, shows that all four scalars are functionally independent.

# 4.4. Using coordinate candidates as coordinates

We now make the obvious choice of the coordinate candidates as the coordinates,

$$t = T, n = N, a = A, b = B (66)$$

where the only coordinate freedom is for t up to an additive constant. We can write down the tetrad vectors immediately in the t, n, a, b coordinates from the respective tables as

$$l^{i} = (0, \frac{-1}{Q}, 0, 0)$$

$$n^{i} = \frac{Q}{a} \left( 1, (S - Ma - \frac{1}{2}a^{2} - \frac{1}{2}b^{2}), n, E \right)$$

$$m^{i} = P(0, 0, -1, -i)$$

$$\bar{m}^{i} = \bar{P}(0, 0, -1, i)$$
(67)

and therefore using (62) the metric is given by,

$$g^{ij} = \begin{pmatrix} 0 & -1/a & 0 & 0\\ -1/a & (-2S + 2Ma + a^2 + b^2)/a & -n/a & -E/a\\ 0 & -n/a & -1 & 0\\ 0 & -E/a & 0 & -1 \end{pmatrix}$$
(68)

where E, M, S are arbitrary functions of the coordinate t. This form now includes the possibility of any of M or E or S being constant and represents the most general conformally flat pure radiation metric.

In fact this metric is slightly different from that obtained by Edgar and Ludwig using the original GHP operators [9]. However if we choose to work with a slightly different spin frame  $\{\mathbf{o}_A, \tilde{\mathbf{I}}_A\}$  and choice of coordinates  $A, B, T, \tilde{N}$ , where

$$\tilde{\mathbf{I}}_A = \mathbf{I}_A + \bar{P}Q\omega o_A$$

$$\tilde{N} = N + \omega$$

then we may show that the two forms are equivalent by choosing  $\omega$  to be a real function of t only which satisfies the condition

$$\dot{\omega} + \omega^2 + M = 0$$

#### 5. Karlhede Classification

In the Karlhede classification one starts by putting the curvature into a canonical form. For the conformally flat pure radiation metrics one may choose  $\mathbf{o}_A$  so that

$$\mathbf{\Phi}_{ABA'B'} = \mathbf{o}_A \mathbf{o}_B \bar{\mathbf{o}}_{A'} \bar{\mathbf{o}}_{B'} \tag{69}$$

and hence  $\Phi = 1$ . This fixes the boost freedom of the spin frame but one still has the rotation and null rotation freedom. Because of the form of the curvature (69), and the Bianchi identities, the only terms one obtains from the derivative of the curvature are  $\tau$  and  $(\gamma + \bar{\gamma})$ . Since  $\rho = \sigma = \kappa = 0$ , then  $\tau$  is invariant under null rotations. However it has non-trivial spin weight and under a rotation

$$\mathbf{o}_A \mapsto \mathrm{e}^{\mathrm{i}\theta/2}\mathbf{o}_A$$

 $\tau$  transforms as

$$\tau \mapsto e^{i\theta} \tau$$

So provided that  $\tau \neq 0$  we may fix the rotation freedom by demanding the  $\tau$  is real. Under a null rotation

$$\iota_A \mapsto \iota_A + \bar{a} \mathbf{o}_A$$

 $(\gamma + \bar{\gamma})$  transforms as

$$(\gamma + \bar{\gamma}) \mapsto (\gamma + \bar{\gamma}) + a(\alpha + \bar{\beta} + \bar{\tau}) + \bar{a}(\bar{\alpha} + \beta + \tau)$$

So that provided  $(\alpha + \bar{\beta} + \bar{\tau}) \neq 0$  one can transform  $(\gamma + \bar{\gamma})$  to zero. However this does not completely fix the null rotation freedom because a one-parameter subgroup

of the (two dimensional) group of null rotations maintains the condition  $(\gamma + \bar{\gamma}) = 0$ . To fix the frame completely one has to go to second order or higher.

In fact the calculations given in this paper show that there is a canonical choice of frame at second order given by choosing the spin and boost freedom so that  $P=1/\sqrt{2}$  and  $Q=\sqrt{A}$ , and the null rotation freedom so that  $P(\bar{P}Q)$  is in canonical form. Note this involves fixing the frame directly at second order and is slightly different from the frame obtained by first reducing the null rotations to a one parameter subgroup at first order and fixing the remaining freedom at second order. In the latter case one has

$$\boldsymbol{\iota}_A = \mathbf{I}_A - \frac{N}{3\sqrt{2}}\mathbf{o}_A$$

Having fixed the frame completely one can calculate scalar invariants by looking at the components of the covariant derivatives of the curvature in the canonical frame. We can extract at most four functionally independent quantities in this way. In the present approach we do not fix the frame, but apply the invariant differential operators to the invariant scalar field A. We then extract zero weighted scalar fields using the gauge dependent quantities P, Q and  $\mathbf{I}_A$ . However these are automatically independent of the frame and are therefore scalar invariants. Further invariants may be obtained by applying the invariant operators to these quantities and extracting zero weighted scalars. In [16], [17] it was shown that all the functional information obtained in the Karlhede classification of type N vacuum spacetimes could be obtained by applying the invariant operators to  $\Psi_4$ . In a similar way, for the conformally flat radiation metrics, all the information may be obtained by applying the differential operators to  $\Phi$ . We give explicit expressions for the terms obtained below.

At zeroth order

$$\Phi = \frac{Q^2}{A} \tag{70}$$

At first order

$$\mathbf{p}\Phi = 0$$

$$\mathbf{\partial}\Phi = \frac{PQ^2}{A^2}$$

$$\mathbf{\partial}'\Phi = \frac{\bar{P}Q^2}{A^2}$$

$$\mathbf{p}'\Phi = -\frac{Q^2}{A^2}(3P\mathbf{I} + 3\bar{P}\bar{\mathbf{I}} + \frac{QN}{A})$$
(71)

Using the fact that  $P\bar{P} = 1/2$  we may solve these for A, P and Q, but I and N are not uniquely determined as one still has left the gauge freedom of a one parameter subgroup of null rotations.

At second order one obtains

$$\mathbf{P}\partial\Phi = 0$$

$$\partial\partial\Phi = \frac{2P^2Q^2}{A^2}$$

$$\partial'\partial\Phi = \frac{Q^2}{A^3}$$

$$\mathbf{P}'\partial\Phi = -\frac{PQ^2}{A^3}(3P\mathbf{I} + 5\bar{P}\bar{\mathbf{I}} + \frac{2QN}{A})$$

$$\mathbf{PP}'\Phi = 0$$

$$\partial\mathbf{P}'\Phi = -\frac{PQ^2}{A^3}(6P\mathbf{I} + 8\bar{P}\bar{\mathbf{I}} + \frac{3QN}{A})$$

$$\mathbf{P'P}'\Phi = \frac{3Q^2}{A^3}\left(4P^2\mathbf{I}^2 + 5P\bar{P}\mathbf{I}\bar{\mathbf{I}} + \bar{P}^2\bar{\mathbf{I}}^2\right) + 12\frac{Q^3N}{A^4}(P\mathbf{I} + \bar{P}\bar{\mathbf{I}})$$

$$+ \frac{Q^4}{A^4}\left(S - 2AM - \frac{5}{2}A^2 + \frac{1}{2}B^2 + 3N^2/A\right)$$

We can now solve these equations for A, N, P, Q and  $\mathbf{I}$  as well as the scalar combination  $S-2AM+\frac{1}{2}B^2$ . Thus at second order we have fixed the frame completely and have three invariant scalar quantities.

At third order we have the equation

$$\partial (S - 2AM + \frac{1}{2}B^2) = P(2M - iB)$$
(73)

so that we can now solve for M, B and S. In section 3.6 we showed that provided M is not constant, then A, B, N and M provide four functionally independent pieces of information, so that generically all the information is obtained at third order. However in the worst case it is possible for all the new terms obtained at third order (namely M, E and S) to be constant. In this case it is necessary to go to 4th order to show that no further relations exist. That such cases can arise in practice was first shown by Koutras [18] who showed that a solution of Wils [19] required the fourth covariant derivative for its invariant classification. Subsequently further examples which required the fourth derivative were found by Edgar and Ludwig [9]. Finally it was shown by Skea [19] (and confirmed by the above calculation) that for all conformally flat pure radiation solutions, all the information about the spacetime is contained in the Riemann tensor and its covariant derivatives to no higher than fourth order.

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## References

- [1] Machado Ramos M P and Vickers J A 1996 Proc. R. Soc. A 450 1–17
- [2] Machado Ramos M P and Vickers J A 1996 Class. Quantum Grav. 13 1579–87
- [3] Geroch R, Held A and Penrose R 1973 J. Math. Phys. 14 874–81
- [4] Held A 1974 Comm. Math. Phys. **37** 311–26
- [5] Held A 1975 Comm. Math. Phys. 44 211–22
- [6] Held A 1974 Gen. Rel. Grav. 7 177–98
- [7] Edgar S B 1992 Gen. Rel. Grav. 24 1267–95
- [8] Edgar S B and Ludwig G 1997 Gen. Rel. Grav. 29 19–59
- [9] Edgar S B and Ludwig G 1997 Gen. Rel. Grav. 29 1309–28
- [10] Karlhede A and Lindström U 1983 Gen. Rel. Grav. 15 597–610
- [11] Bradley M and Karlhede A 1990 Class. Quantum Grav. 7 449–463
- [12] Bradley M and Marklund M 1996 Class. Quantum Grav. 13 3021–37
- [13] Marklund M 1997 Class. Quantum Grav. 14 1267–84
- [14] Marklund M and Bradley M 1998 Preprint
- [15] Machado Ramos M P 1997PhD Thesis, University of Southampton
- [16] Machado Ramos M P and Vickers J A 1996 Class. Quantum Grav. 13 1589–99
- [17] Machado Ramos M P 1998 Class. Quantum Grav. 15 435-43
- [18] Koutras A 1992 Class. Quantum Grav. 9 L143-5
- [19] Skea J E F 1997 Class. Quantum Grav. 14 2393-404